## 3.1: PERTURBATION THEOREMS FOR WAVEGUIDE JUNCTIONS, WITH APPLICATIONS

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Perturbation theorems. - Our perturbation theorems [1,2] are stated in the context of a theory of waveguide junctions [3].

A waveguide junction is a linear electromagnetic system possessing ideal waveguide leads and is subject to excitation only through non-attenuated modes in these leads. The domain of the electromagnetic field is the finite region V; the surface S, the complete boundary of V, consists of a part $S_o$, coinciding with a perfectly conducting surface, and the parts $S_1$, $S_2$, . . . , $S_n$, where $S_m$ is the terminal surface in the mth of the n waveguide leads. Within V, the complex vectors $\underline{E}$, $\underline{H}$ of the time-harmonic electromagnetic field satisfy Maxwell's equations, which are written $\underline{E} = \mathcal{E}(\underline{H})$, $H = \mathcal{H}(E)$, using the operators

$$\mathcal{E} = (j\omega\varepsilon)^{-1} \cdot \nabla X, \qquad \mathcal{H} = -(j\omega\mu)^{-1} \cdot \nabla X. \tag{1}$$

Here j is the imaginary unit, $\omega/(2\pi)$ is the frequency, and $\mu$, $\varepsilon$ are in general complex nonsymmetric dyadic point-functions, which reduce to real scalar constants in the ideal portions of the waveguides.

The tangential components $\underline{E}_t$, $\underline{H}_t$ of $\underline{E}$, $\underline{H}$ on $S_m$ are expressible in the form

$$\underline{E}_t = \sum_{\mu=1}^{\nu_m} v_{m\mu} \underline{e}^o_{m\mu} , \qquad \underline{H}_t = \sum_{\mu=1}^{\nu_m} i_{m\mu} \underline{e}^o_{m\mu} . \tag{2}$$

Here $\nu_m$ is the number of propagated modes in the mth waveguide, $v_{m\mu}$ and $i_{m\mu}$ are scalar coefficients, and the terminal basis-fields $\underline{e}^o_{m\mu}$ and $\underline{h}^o_{m\mu}$ are real and subject to

$$\int_{S_m} \underline{e}^o_{m\mu} \underline{h}^o_{m\lambda} \underline{n}_m \, dS = \delta_{\mu\lambda} , \tag{3}$$

where $\delta_{\mu\lambda}$ is a Kronecker delta and $\underline{n}_m$ denotes $\underline{n}$ on $S_m$; here and subsequently integrands in surface integrals are scalar triple products. In what follows, single-letter indices p, q, . . . will be used to indicate both waveguide and mode.

On $S_Q$, the boundary condition $\underline{n} \times \underline{E} = 0$ applies. The additional prescription $v_q = \delta_{qp}$ for a given p determines an electromagnetic field in V, which is denoted $\underline{e}_p, \mathcal{H}(\underline{e}_p)$. Alternatively, the prescription $i_q = \delta_{qp}$ for given p determines an electromagnetic field in V; this is denoted[3] $\mathcal{E}(\underline{h}_p)$, $\underline{h}_p$. It can be shown that

$$Z_{pq} = \int_S \mathcal{E}(\underline{h}_q) \, \underline{h}_p \, \underline{n} \, dS , \qquad Y_{pq} = \int_S \underline{e}_p \mathcal{H}(\underline{e}_q) \, \underline{n} \, dS, \tag{4}$$

where $Z_{pq}$ and $Y_{pq}$ are the elements of the impedance and admittance matrices relating the v's and the i's.

In addition to an "original" system having the parameters $\mu$, $\varepsilon$, we must consider not only a "changed" system having parameters $\mu'$, $\varepsilon'$, but also the systems "adjoint" to the original system and to the changed system, respectively. (By the "adjoint" of a given system is meant one having parameters $\hat{\mu}$, $\hat{\varepsilon}$ equal respectively to the transposes of the $\mu$, $\varepsilon$ of the given system. If $\mu$ and $\varepsilon$ are symmetric, the system is "self-adjoint"). The region V and the terminal basis-fields are to be the same for all four systems. Quantities associated with the changed and with the adjoint systems are distinguished throughout by primes and circumflexes, respectively.

The immittance matrix elements of a given system and its adjoint satisfy a reciprocity relation[4], e.g., $Z_{pq} = \hat{Z}_{qp}$. Using the reciprocity relation, combining suitable expressions of the type (4), and converting to a volume integral, one finds

$$Z'_{pq} - Z_{pq} = j\omega \int_{V'} \left[ \hat{\underline{h}}'_p \cdot (\mu' - \mu) \cdot \underline{h}_q - \hat{\mathcal{E}}'(\hat{\underline{h}}'_p) \cdot (\varepsilon' - \varepsilon) \cdot \mathcal{E}(\underline{h}_q) \right] dV \quad . \qquad (5)$$

The region of integration V' is the subregion of V in which one or both of the primed parameters differ from the unprimed ones. Equation (5) is one form of perturbation theorem. We also have

$$Z'_{pq} - Z_{pq} = \int_{S'} \left[ \hat{\mathcal{E}}'(\hat{\underline{h}}'_p)\underline{h}_q - \mathcal{E}(\underline{h}_q)\hat{\underline{h}}'_p \right] \underline{n}\, dS \quad , \qquad (6)$$

where S' is the boundary of V'. The admittance analog of (6) is

$$Y'_{pq} - Y_{pq} = \int_{S'} \left[ \underline{e}_q\, \hat{\mathcal{H}}'(\hat{\underline{e}}'_p) - \hat{\underline{e}}'_p\, \mathcal{H}(\underline{e}_q) \right] \underline{n}\, dS \quad . \qquad (7)$$

The surface-integral expressions are presumed applicable (and are applied) in some cases where a limiting procedure would be required to give meaning to the volume-integral expressions--e. g., when conductivity is allowed to become infinite in V'.

Applications. - The results obtained in the first and second examples below are believed to be new, useful, and correct to the lowest order in the perturbation parameters involved; the significance of the third will be discussed.

(1) Finitely conducting half-round inductive obstacles in finitely conducting rectangular waveguide. The geometry and some of the notation are shown in Figures 1 and 2. Reference planes for the structure coincide at z = 0. The perturbed electric field is estimated from the unperturbed magnetic field using the surface-impedance approximation; the unperturbed magnetic field is obtained to the desired approximation with the aid of previous work on the perfectly conducting case [5]. One finds

$$Z'_{ee} = Z_{ee} \left[ 1 + (1 - j)\frac{\mu_m}{\mu}\left(\frac{3\pi - 8}{\pi}\right)\frac{\delta}{R} \right] \quad , \qquad (8)$$

where $\mu$ and $\mu_m$ respectively denote the permeability in the interior of the waveguide and in the "metal," and $\delta$ is the skin depth in the metal. For (8) to be a "good" approximation it is necessary that $\delta << R << a$. ($Z'_{oo}$ is not calculated since it is not needed in obtaining the lowest-order correction for the matched-termination reflection coefficient.)

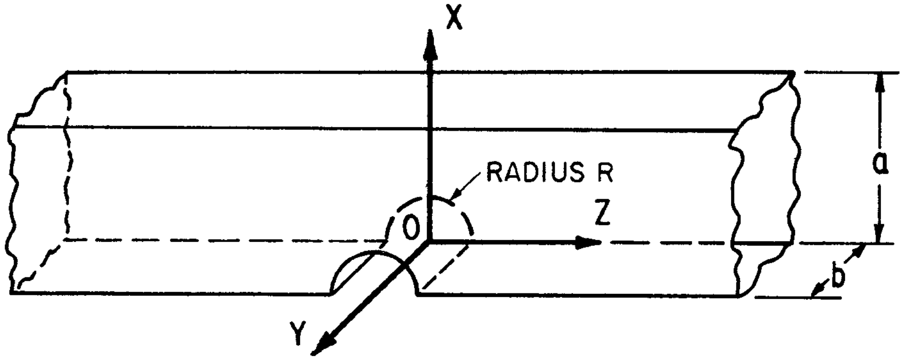


Fig. 1. Half-round inductive obstacle.

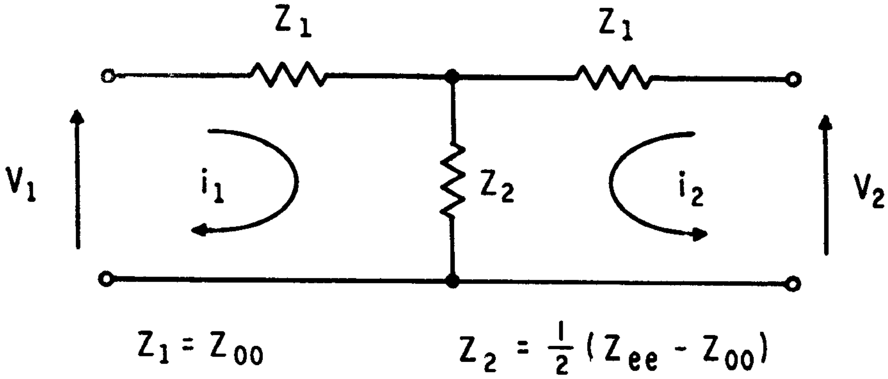


Fig. 2. Equivalent network for half-round obstacle.

(2) Junction of rectangular with filleted waveguide. Geometry and geometrical parameters are shown in Figure 3. Perfect conductivity is assumed. The unperturbed electromagnetic field is that of a suitably normalized $TE_{10}$-mode of the rectangular guide, traveling in the +z-direction. The integral (6) is evaluated, using the unperturbed magnetic field as an approximation for the perturbed one, and using the familiar artifice of assuming an infinitesimal attenuation to secure convergence. From the impedance given by (6) one finds for the reflection coefficient at the junction

$$S_{11} = \left(\frac{\lambda g R}{a}\right)^2 \frac{4 - \pi}{8ab}$$

where λg is the (unperturbed) guide wavelength.

(3) Perfectly conducting half-round inductive obstacles as a perturbation of perfectly conducting rectangular waveguide. Figures 1 and 2 are again pertinent. In this example it turns out, contrary to what one might expect on fairly general grounds, that the expedient of

approximating perturbed magnetic fields by unperturbed ones leads to results that are distinctly not correct lowest-order values.

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1. Cf. G. D. Monteath, "Application of the compensation theorem to certain radiation and propagation problems," Proc. IEE, Part IV, 98, 23-30 (1951).

2. Cf. Alfred G. Redfield, "An electrodynamic perturbation theorem, with application to nonreciprocal systems," J. Appl. Phys. 25, 1021-1024 (1954).

3. Adapted from D. M. Kerns, "Analysis of symmetrical waveguide junctions," J. Res. NBS 46, 267-282 (1951).

4. Stated by Rumsey and attributed to M. H. Cohen. V. H. Rumsey, "The reaction concept in electromagnetic theory," Phys. Rev. 94, 1483-1491 (1954). Errata, Phys. Rev. 95, 1705 (1954).

5. D. M. Kerns, "Half-round inductive obstacles in rectangular wave-guide," J. Res. NBS 64B, 113-130 (1960).

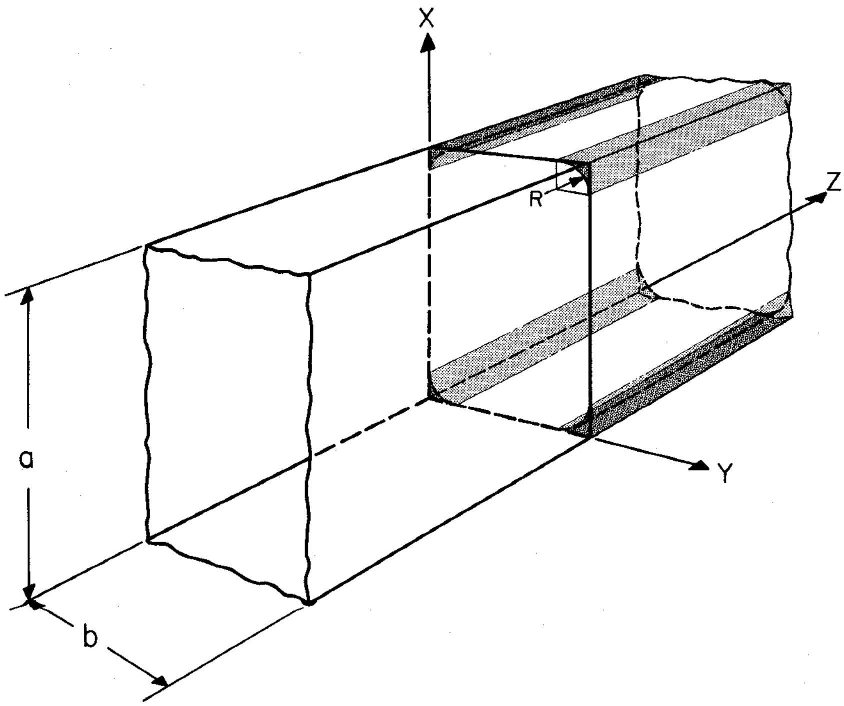


Fig. 3. Junction of rectangular and filleted waveguide.